SECTION 3.8: IMPLICIT DIFFERENTIATION

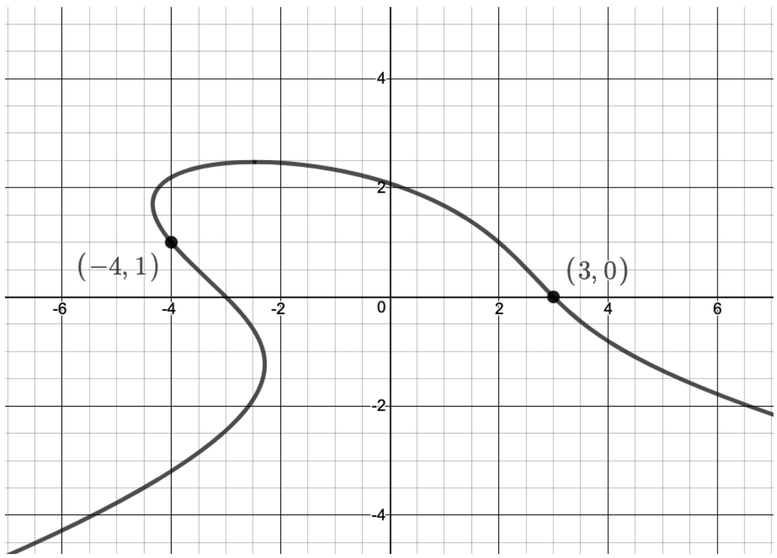
RECALL: Given two variables x and y, we say 'y is a function of x' to mean that for each choice of x, there is (at most) one corresponding value of y.

We may express this relationship **explicitly** if we can solve for y in terms of x. That is, if we can write: $y = f(x)$.

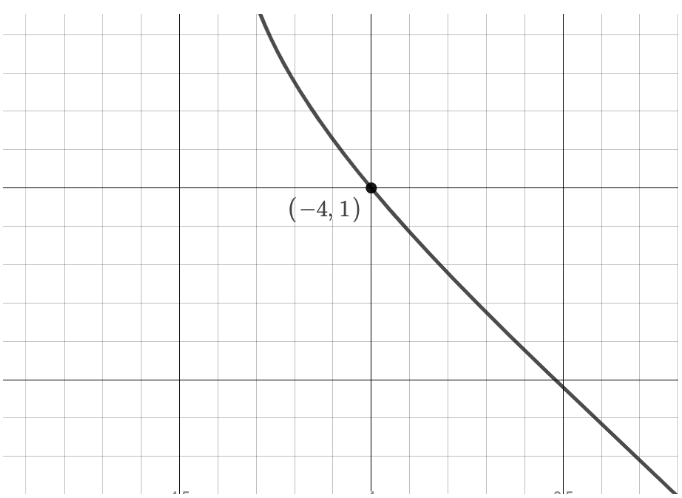
For example, $y = x^2 - 3x + 1$ explicitly shows y is a function of x.

However, we may also describe this relationship **implicitly** using an equation such as $x^2 + 2xy + y^3 = 9$. In this case, it is (inconvenient, to say the least) to solve for y in terms of x.

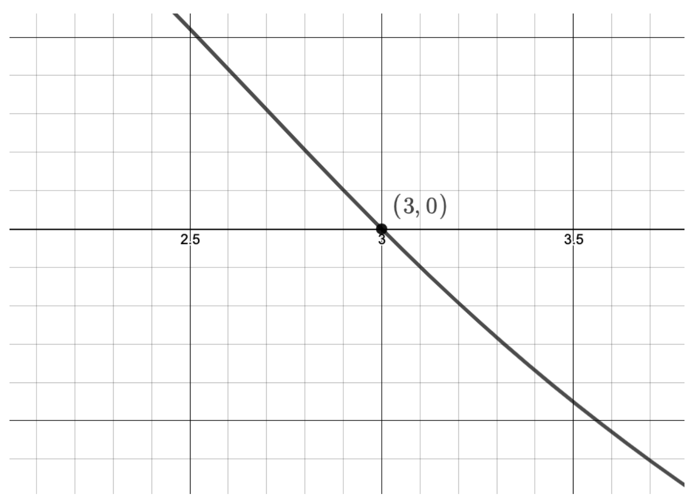
Moreover, the graph of this equation fails the vertical line test, so taken as a whole, this equation doesn't even describe y as a function of x!



That being said, we still may want to know information about the slope of this curve. For instance, if we 'zoom in' near the point $(4, -1)$ and again near the point $(3, 0)$, there is a window in which the graph **does** pass the vertical line test. That is to say, **locally**, y is a function of x. And since derivatives are all about 'local linearity' it stands to reason we should be able to find tangent lines at these points.



'near' $(4, -1)$



'near' $(3, 0)$

What we need is a way to find $\dfrac{dy}{dx}$ **without** having to solve for y in terms of x first.

This is the concept of **implicit differentiation**.

We start with equation: $x^2 + 2xy + y^3 = 9$ and differentiate both sides with respect to x as follows:

$$\begin{aligned} D_x\left[x^2 + 2xy + y^3\right] &= D_x[9] \\ D_x\left[x^2\right] + 2D_x[xy] + D_x\left[y^3\right] &= 0 \end{aligned}$$

Using the power rule, we know $D_x\left[x^2\right] = 2x$. But how do we simplify $D_x[xy]$ and $D_x\left[y^3\right]$?

Since we're thinking of y locally as a function of x, let's write $y = f(x)$.

Then $D_x[xy] = D_x[x\,f(x)]$ so we know to use the product rule:

$$D_x[x\,f(x)] = D_x[x]f(x) + x\,D_x[f(x)] = (1)f(x) + x\,f'(x) = f(x) + x\,f'(x)$$

Hence using the original notation with $y = f(x)$, we get:

$$D_x[xy] = D_x[x]\,y + x\,D_x[y] = y + x\,\frac{dy}{dx}$$

To find $D_x\left[y^3\right]$, we write $y = f(x)$ and use the Power Rule: $D_x\left[y^3\right] = D_x\left[(f(x))^3\right] = 3(f(x))^2 f'(x)$. Hence:

$$D_x\left[y^3\right] = 3y^2\,\frac{dy}{dx}$$

So, we continue:

$$\begin{aligned} D_x\left[x^2 + 2xy + y^3\right] &= D_x[9] \\ D_x\left[x^2\right] + 2D_x[xy] + D_x\left[y^3\right] &= 0 \\ D_x\left[x^2\right] + 2D_x[xy] + D_x\left[y^3\right] &= 0 \\ 2x + 2\left(D_x[x]\,y + x\,D_x[y]\right) + 3y^2\,\frac{dy}{dx} &= 0 \\ 2x + 2\left(y + x\,\frac{dy}{dx}\right) + 3y^2\,\frac{dy}{dx} &= 0 \\ 2x + 2y + 2x\,\frac{dy}{dx} + 3y^2\,\frac{dy}{dx} &= 0 \\ 2x\,\frac{dy}{dx} + 3y^2\,\frac{dy}{dx} &= -2x - 2y \\ \frac{dy}{dx}\left(2x + 3y^2\right) &= -2x - 2y \\ \frac{dy}{dx} &= \frac{-2x - 2y}{2x + 3y^2} \end{aligned}$$

EXAMPLE 1: Write the equation of the tangent line to the curve $x^2 + 2xy + y^3 = 9$ at the point $(-4, 1)$ using:

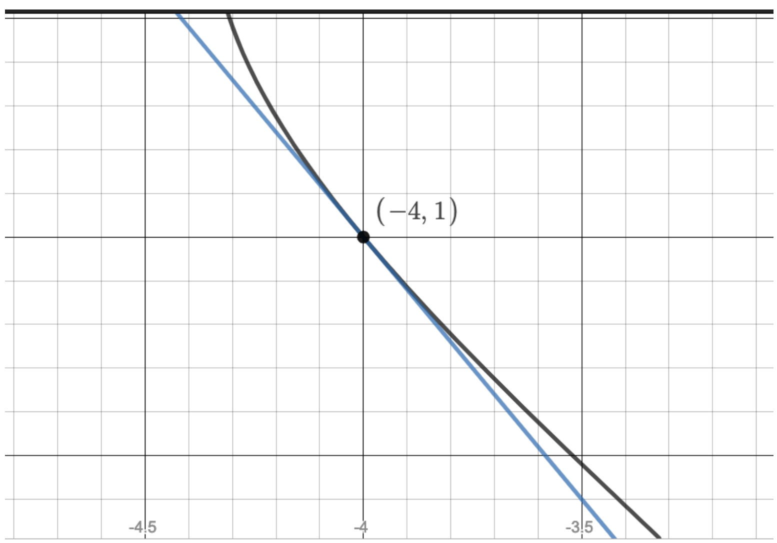
$$\frac{dy}{dx} = \frac{-2x - 2y}{2x + 3y^2}$$

Check your answer using a graphing utility.

To find the slope of the tangent line at $(-4, 1)$, we substitute $x = -4$ and $y = -1$ into the formula for $\frac{dy}{dx}$:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(-4,1)} = \frac{-2(-4) - 2(1)}{2(-4) + 3(1)^2} = -\frac{6}{5}$$

Hence, the tangent line is: $y = -\frac{6}{5}(x - (-4)) + 1$ which simplifies to: $y = -\frac{6}{5}x - \frac{19}{5}$. Graphically, we have:



EXAMPLE 2: (VIDEO) Write the equation of the tangent line to $x^2 + 2xy + y^3 = 9$ at the point $(3, 0)$ using:

$$\frac{dy}{dx} = \frac{-2x - 2y}{2x + 3y^2}$$

Check your answer using a graphing utility.

To find the slope of the tangent line at $(3, 0)$, we substitute $x = 3$ and $y = 0$ into the formula for $\frac{dy}{dx}$:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(3,0)} = \frac{-2(3) - 2(0)}{2(3) + 3(0)^2} = -1$$

Hence, the tangent line is: $y = (-1)(x - 3) + 0$ which simplifies to: $y = -x + 3$.

EXAMPLE 3: Consider the Unit Circle: $x^2 + y^2 = 1$.

1. Use implicit differentiation to find a formula for $\dfrac{dy}{dx}$ in terms of x and y.

   We begin differentiating both sides of $x^2 + y^2 = 1$ with respect to x:

$$
\begin{aligned}
D_x\left[x^2 + y^2\right] &= D_x[1] \\
D_x\left[x^2\right] + D_x\left[y^2\right] &= 0 \\
2x + 2y\,\frac{dy}{dx} &= 0 \\
2y\,\frac{dy}{dx} &= -2x \\
\frac{dy}{dx} &= -\frac{2x}{2y} \\
\frac{dy}{dx} &= -\frac{x}{y}, \quad \text{provided } y \neq 0
\end{aligned}
$$

2. Use implicit differentiation to find a formula for $\dfrac{d^2y}{dx^2}$ in terms of x and y.

   Recall $\dfrac{d^2y}{dx^2}$ means to find the **second** derivative, Hence:

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= D_x\left[\frac{dy}{dx}\right] \\
&= D_x\left[-\frac{x}{y}\right] \\
&= -\frac{y\,D_x[x] - x\,D_x[y]}{y^2} && \text{Quotient Rule} \\
&= -\frac{y\,(1) - x\,\dfrac{dy}{dx}}{y^2} \\
&= -\frac{y - x\left(-\dfrac{x}{y}\right)}{y^2} && \text{Since } \frac{dy}{dx} = -\frac{x}{y} \\
&= -\frac{y + \dfrac{x^2}{y}}{y^2} \\
&= -\frac{y^2 + x^2}{y^3} && \text{clear the complex fraction.} \\
\frac{d^2y}{dx^2} &= -\frac{1}{y^3} && \text{Since } x^2 + y^2 = 1.
\end{aligned}
$$

EXAMPLE 4: (VIDEO) Consider the curve described by the equation: $x^3 \sin(2y) = 3x + y$.

1. Find an expression for $\frac{dy}{dx}$.

Ans: $\frac{dy}{dx} = \frac{3 - 3x^2 \sin(2y)}{2x^3 \cos(2y) - 1}$

2. Write the equation of the tangent line at $(0, 0)$.

   Check your answer using a graphing utility.

Ans: $y = -3x$

EXAMPLE 5: (VIDEO) Consider the curve described by the equation: $8x^3 + 16xy + y^3 = 0$

1. Find an expression for $\frac{dy}{dx}$.

Ans: $\frac{dy}{dx} = \frac{-24x^2 - 16y}{16x + 3y^2}$

2. Write the equation of the tangent line at $(-2, -4)$.

Ans: $y = -2x - 8$

3. **CHALLENGE:** Find all points on the curve with a horizontal tangent line.

   To have a horizontal tangent line, we need $\frac{dy}{dx} = 0$. Hence, we need $\frac{-24x^2 - 16y}{16x + 3y^2} = 0$ so $-24x^2 - 16y = 0$.

   This means $16y = -24x^2$ or, after solving for y, $y = -\frac{3}{2}x^2$.

   Hence, we need to find the points on the curve $8x^3 + 16xy + y^3 = 0$ which satisfy $y = -\frac{3}{2}x^2$.

   We substitute $y = -\frac{3}{2}x^2$ into $8x^3 + 16xy + y^3 = 0$ and solve for x:

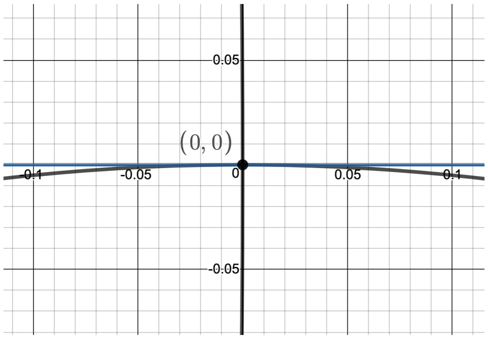
$$
\begin{aligned}
8x^3 + 16x\left(-\frac{3}{2}x^2\right) + \left(\frac{3}{2}x^2\right)^3 &= 0 \\
8x^3 - 24x^3 - \frac{27}{8}x^6 &= 0 \\
-16x^3 - \frac{27}{8}x^6 &= 0 \\
-x^3\left(16 + \frac{27}{8}x^3\right) &= 0
\end{aligned}
$$

We get $-x^3 = 0$, so $x = 0$ or $16 + \frac{27}{8}x^3 = 0$ which gives $x^3 = -\frac{128}{27}$ or $x = -\frac{4\sqrt[3]{2}}{3}$.

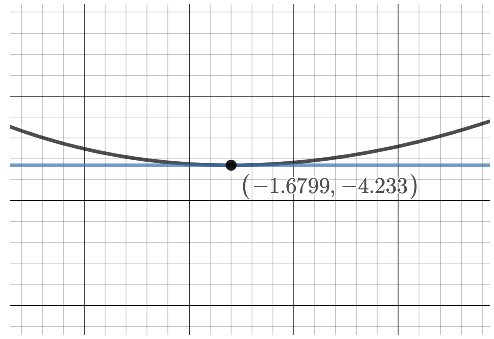
Since $y = -\frac{3}{2}x^2$, $x = 0$ gives $y = -\frac{3}{2}(0)^2 = 0$ so one point is $(0,0)$.

For $x = -\frac{4\sqrt[3]{2}}{3}$, $y = -\frac{3}{2}\left(-\frac{4\sqrt[3]{2}}{3}\right)^2 = -\frac{8\sqrt[3]{4}}{3}$ so the other point is $\left(-\frac{4\sqrt[3]{2}}{3}, -\frac{8\sqrt[3]{4}}{3}\right)$.

Graphically:



'near' $(0,0)$



'near' $\left(-\frac{4\sqrt[3]{2}}{3}, -\frac{8\sqrt[3]{4}}{3}\right)$

NOTE: We have a vertical tangent, $x = 0$ at $(0,0)$ as well!

EXPLORATION: (VIDEO) For the curve $8x^3 + 16xy + y^3 = 0$, find an expression for $\dfrac{dx}{dy}$ implicitly.

That is, treat y as the independent variable and $x = f(y)$ as the dependent variable.

$$\begin{aligned}
D_y\left[8x^3 + 16xy + y^3\right] &= D_y[0] \\
D_y\left[8x^3\right] + D_y[16xy] + D_y\left[y^3\right] &= 0 \\
24x^2 D_y[x] + 16 D_y[xy] + 3y^2 &= 0 \\
24x^2 \frac{dx}{dy} + 16\left(D_y[x]\, y + x\, D_y[y]\right) + 3y^2 &= 0 \\
24x^2 \frac{dx}{dy} + 16\left(\frac{dx}{dy} y + x(1)\right) + 3y^2 &= 0 \\
24x^2 \frac{dx}{dy} + 16y \frac{dx}{dy} + 16x + 3y^2 &= 0 \\
24x^2 \frac{dx}{dy} + 16y \frac{dx}{dy} &= -16x - 3y^2 \\
\left(24x^2 + 16y\right) \frac{dx}{dy} &= -16x - 3y^2 \\
\frac{dx}{dx} &= \frac{-16x - 3y^2}{24x^2 + 16y}
\end{aligned}$$

From above, we have $\dfrac{dy}{dx} = \dfrac{-24x^2 - 16y}{16x + 3y^2} = -\dfrac{24x^2 + 16y}{16x + 3y^2}$ and we have $\dfrac{dx}{dy} = \dfrac{-16x - 3y^2}{24x^2 + 16y} = -\dfrac{16x + 3y^2}{24x^2 + 16y} = \dfrac{1}{\frac{dy}{dx}}$

HOMEWORK: Section 3.8: 5 - 55 odd, 63, 83*, 91*